CM0832 Elements of Set Theory
6. Cardinal Numbers and the Axiom of Choice

Andrés Sicard-Ramírez

Universidad EAFIT

Semester 2017-2

Preliminaries

Textbook

Enderton (1977). Elements of Set Theory.

Convention

The numbers and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook.

Equinumerosity

Observation

A one-to-one function from A onto B is called a **one-to-one correspondence** between A and B.

Equinumerosity

Observation

A one-to-one function from A onto B is called a **one-to-one correspondence** between A and B.

Definition

A set A is **equinumerous** to a set B, denoted $A \approx B$, iff there is a one-to-one correspondence between A and B.

Equinumerosity

Observation

A one-to-one function from A onto B is called a **one-to-one correspondence** between A and B.

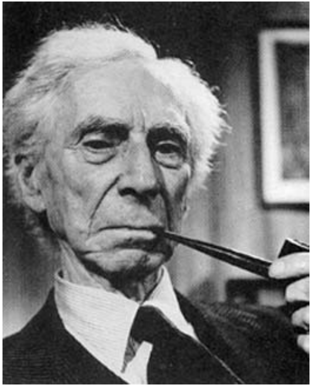
Definition

A set A is **equinumerous** to a set B, denoted $A \approx B$, iff there is a one-to-one correspondence between A and B.

Example

Whiteboard.

Equinumerosity



'The possibility that whole and part may have the same number of terms is, it must be confessed, shocking to common-sense.' (Russell 1903, p. 358)

Equinumerosity

Theorem 6B(a)

The set ω is not equinumerous to the set $\mathbb{R}$ of real numbers.

Equinumerosity

Proof.

Let's suppose $\omega \approx \mathbb{R}$, that is, there is an one-to-one correspondence $f : \omega \to \mathbb{R}$ such that[†]

$$
\begin{aligned}
f(0) &= 236.001\ldots, \\
f(1) &= -7.777\ldots, \\
f(2) &= 3.1415\ldots, \\
&\vdots
\end{aligned}
$$

(continued on next slide)

[†] 'We assume that a decimal expansion does not contain only the digit 9 from some place on, so each real number has a unique decimal expansion.' (Hrbacek and Jech [1978] 1999, Theorem 6.1, p. 90)

Equinumerosity

Proof.

Let $x = 0.d_1 d_2 d_3 \ldots \in \mathbb{R}$, where

$$d_{n+1} = \begin{cases} 4, & \text{if } f(n) \neq 4; \\ 5, & \text{if } f(n) = 4. \end{cases}$$

The number x does not belong to the above enumeration. Therefore, $\mathbb{R}$ is non-enumerable.

∎

On Refutations of Cantor's Diagonal Argument

'I dedicate this essay to the two-dozen-odd people whose refutations of Cantor's diagonal argument have come to me either as referee or as editor in the last twenty years or so... A few years ago it occurred to me to wonder why so many people devote so much energy to refuting this harmless little argument—what had it done to make them angry with it?... These pages report the results.' (Hodges 1998, p. 1)

Finite Sets

Definition

A set is **finite** iff it is equinumerous to some natural number. Otherwise it is **infinite**.

Finite Sets

Definition

A set is **finite** iff it is equinumerous to some natural number. Otherwise it is **infinite**.

Corollary 6C

No finite set is equinumerous to a proper subset of itself.

Finite Sets

Definition

A set is **finite** iff it is equinumerous to some natural number. Otherwise it is **infinite**.

Corollary 6C

No finite set is equinumerous to a proper subset of itself.

Corollary 6D

(i) Any set equinumerous to a proper subset of itself is infinite.

Finite Sets

Definition

A set is **finite** iff it is equinumerous to some natural number. Otherwise it is **infinite**.

Corollary 6C

No finite set is equinumerous to a proper subset of itself.

Corollary 6D

(i) Any set equinumerous to a proper subset of itself is infinite.

(ii) The set ω is infinite.

The Continuum Hypothesis

The continuum hypothesis (CH)

There is no a set whose cardinality is strictly between the cardinality of the set of the natural numbers and the cardinality of the set of real numbers, that is,

$$2^{\aleph_0} = \aleph_1.$$

CH could not be disproved (Gödel 1938) nor proved (Cohen 1963) in ZFC, that is, CH is independent of ZFC set theory.

References

Paul J. Cohen (1963). The Independence of the Continuum Hypothesis. Proceedings of the National Academy of Sciences of the United States of America 50.6, pp. 1143–1148. DOI: 10.1073/pnas.50.6.1143 (cit. on p. 15).

Herbert B. Enderton (1977). Elements of Set Theory. Academic Press (cit. on p. 2).

Kurt Gödel (1938). The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis. Proceedings of the National Academy of Sciences of the United States of America 24.12, pp. 556–557. DOI: 10.1073/pnas.24.12.556 (cit. on p. 15).

Wilfrid Hodges (1998). An Editor Recalls some Hopeless Papers. The Bulletin of Symbolic Logic 4.1, pp. 1–16. DOI: 10.2307/421003 (cit. on p. 10).

Karel Hrbacek and Thomas Jech [1978] (1999). Introduction to Set Theory. Third Edition, Revised and Expanded. Marcel Dekker (cit. on p. 8).

Bertrand Russell (1903). The Principles of Mathematics. Cambridge University Press (cit. on p. 6).