CM0081 Formal Languages and Automata
§ 3.1 Regular Expressions

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Semester 2024-1

Preliminaries

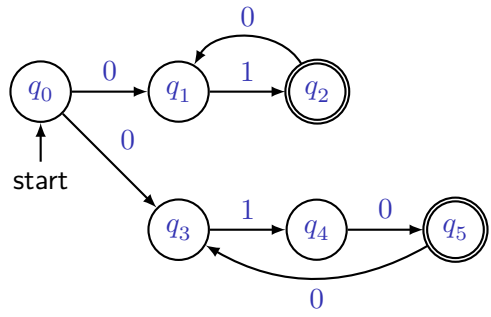
Conventions

- The number and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook [Hopcroft, Motwani and Ullman (1979) 2007].
- The natural numbers include the zero, that is, $\mathbb{N} = \{0, 1, 2, \dots\}$.
- The power set of a set A, that is, the set of its subsets, is denoted by $\mathcal{P}A$.

Introduction: Description of regular languages

$$(\mathbf{01})(\mathbf{01})^* + (\mathbf{010})(\mathbf{010})^*$$

Algebraic description



Machine-like description

Introduction: Regular Expressions

Features

- Algebraic description of regular languages

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Uses

- Search commands (e.g. GREP)
- Lexical-analyzer generators (e.g. LEX and ALEX)
- Domain specific languages (DSLs)

Operations on Languages

Notation

The power set of a set A is denoted $\mathcal{P}A$.

Operations on Languages

Definition

Let L, L_1 and L_2 be languages on an alphabet Σ.

(i) **Union** of languages:

$$\cup : \mathcal{P}\Sigma^* \times \mathcal{P}\Sigma^* \to \mathcal{P}\Sigma^*$$
$$L_1 \cup L_2 := \{\, x \mid x \in L_1 \text{ or } x \in L_2 \,\}.$$

(ii) **Concatenation** of languages:

$$\cdot : \mathcal{P}\Sigma^* \times \mathcal{P}\Sigma^* \to \mathcal{P}\Sigma^*$$
$$L_1 \cdot L_2 := \{\, x \cdot y \mid x \in L_1 \text{ and } y \in L_2 \,\}.$$

(iii) **Powers** of a language:

$$(-)^{(-)} : \mathcal{P}\Sigma^* \times \mathbb{N} \to \mathcal{P}\Sigma^*$$
$$L^0 := \{\varepsilon\},$$
$$L^{n+1} := L \cdot L^n.$$

(iv) **Kleene closure** of a language:

$$(-)^* : \mathcal{P}\Sigma^* \to \mathcal{P}\Sigma^*$$
$$L^* := \bigcup_{n \geq 0} L^n.$$

Operations on Languages

Examples

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Operations on Languages

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- If $L = \{0, 1\}$, then L^* consists of all strings of 0's and 1's and the empty word.
- If $L = \{\, 0^n \mid n \geq 1 \,\}$, then $L^* = L \cup \{\varepsilon\}$.
- If $L = \{0, 11\}$, then L^* consists of the empty word and those strings of 0's and 1's such that the 1's come in pairs.

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- If $L = \{0, 11\}$, then L^* consists of the empty word and those strings of 0's and 1's such that the 1's come in pairs.
- Powers on $\emptyset$

$$\begin{aligned} \emptyset^0 &= \{\varepsilon\}, \\ \emptyset^i &= \emptyset, \quad \text{for } i \geq 1, \\ \emptyset^* &= \{\varepsilon\}. \end{aligned}$$

What the Regular Expressions Are

Definition

Let Σ be an alphabet. The **regular expressions** (regex's) on Σ are inductively defined by:

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  - (iii) If $a \in \Sigma$ then $\boldsymbol{a}$ is a regex.
- Inductive step
  If E and F are regex's then
  - (i) $E + F$ is a regex,
  - (ii) $E \cdot F$ is a regex,
  - (iii) E^* is a regex and
  - (iv) (E) is a regex.

Precedence of Operators

Order of precedence and associative

Precedence from highest to lowest: $()$, *, $\cdot$ and $+$.

Associative: The operators $\cdot$ and $+$ are left-associative.

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Example

$$\begin{aligned} \mathbf{01}^* + \mathbf{1} &= (\mathbf{0}(\mathbf{1}^*)) + \mathbf{1} \\ &\neq (\mathbf{01})^* + \mathbf{1} \\ &\neq \mathbf{0}(\mathbf{1}^* + \mathbf{1}) \end{aligned}$$

Languages Denoted by Regular Expressions

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Let E be a regular expression. The **language denoted** by E, denoted by $\mathrm{L}(E)$, is inductively defined by:

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- Inductive step

Let $\mathrm{L}(E)$ and $\mathrm{L}(F)$ be the languages denoted by the regular expressions E and F, then

$$\mathrm{L}(E + F) := \mathrm{L}(E) \cup \mathrm{L}(F),$$
$$\mathrm{L}(E \cdot F) := \mathrm{L}(E) \cdot \mathrm{L}(F),$$
$$\mathrm{L}(E^*) := (\mathrm{L}(E))^*,$$
$$\mathrm{L}((E)) := \mathrm{L}(E).$$

Languages Denoted by Regular Expressions

Example

E	$\mathrm{L}(E)$
$\boldsymbol{a} + \boldsymbol{b}$	$\mathrm{L}(\boldsymbol{a}) \cup \mathrm{L}(\boldsymbol{b}) = \{a\} \cup \{b\} = \{a, b\}$

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$\boldsymbol{a}^*$	$\{\varepsilon, a, aa, aaa, ...\}$

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$\boldsymbol{a} + (\boldsymbol{ab})^*$	$\{a, \varepsilon, ab, abab, ababab, ...\}$
$(\mathbf{0} + \mathbf{1})^*\mathbf{01}(\mathbf{0} + \mathbf{1})^*$	$\{\, x01y \mid x, y \in \{0, 1\}^* \,\}$

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$\boldsymbol{a} + (\boldsymbol{ab})^*$	$\{a, \varepsilon, ab, abab, ababab, ...\}$
$(\mathbf{0} + \mathbf{1})^*\mathbf{01}(\mathbf{0} + \mathbf{1})^*$	$\{\, x01y \mid x, y \in \{0, 1\}^* \,\}$
$\boldsymbol{a_i}(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \cdots + \boldsymbol{a_n})^*$	$\{\, w \in \Sigma^* \mid w$ starts by $a_i \,\}$

Languages Denoted by Regular Expressions

Example

Write a regular expression for the language L defined by

$$L = \{\, w \in \{0,1\}^* \mid 0 \text{ and } 1 \text{ alternate in } w \,\}.$$

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Solution.

$$(\mathbf{01})^* + (\mathbf{10})^* + \mathbf{0}(\mathbf{10})^* + \mathbf{1}(\mathbf{01})^*$$

Languages Denoted by Regular Expressions

Example

Write a regular expression for the language L defined by

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Solution.

$$(\mathbf{01})^* + (\mathbf{10})^* + \mathbf{0}(\mathbf{10})^* + \mathbf{1}(\mathbf{01})^*$$

Other solution.

$$(\varepsilon + \mathbf{1})(\mathbf{01})^*(\varepsilon + \mathbf{0})$$

Languages Denoted by Regular Expressions

Example

The regular expression

$$(\mathbf{10} + \mathbf{0})^*(\varepsilon + \mathbf{1})$$

denotes the set of strings of 0's and 1's that have no two adjacent 1's.

Languages Denoted by Regular Expressions

Example

Write a regular expression for denoting the set of strings over $\Sigma = \{0, 1\}$ not ending in 01.

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Solution.

$$\varepsilon + \mathbf{0} + \mathbf{1} + (\mathbf{0} + \mathbf{1})^*(\mathbf{00} + \mathbf{10} + \mathbf{11})$$

Derivatives of Regular Expressions

Observation

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Definition

Let $L \subseteq \Sigma^*$ be a language and $a \in \Sigma$ a symbol. We define the **derivative** of L by a, denoted by $\partial_a L$, by

$$\partial_a : \mathcal{P}\Sigma^* \to \mathcal{P}\Sigma^*$$
$$\partial_a L = \{\, x \in \Sigma^* \mid ax \in L \,\}.$$

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$$\partial_a : \mathcal{P}\Sigma^* \to \mathcal{P}\Sigma^*$$
$$\partial_a L = \{\, x \in \Sigma^* \mid ax \in L \,\}.$$

Example

$$\partial_a\{abab, abba\} = \{bab, bba\},$$
$$\partial_a \mathrm{L}(\boldsymbol{ab}^*) = \mathrm{L}(\boldsymbol{b}^*),$$
$$\partial_b \mathrm{L}(\boldsymbol{ab}^*) = \emptyset.$$

Derivatives of Regular Expressions

Definition

Let E be a regular expression on Σ and let $a \in \Sigma$ be a symbol. We define recursively the **derivative** of E by a, denoted $\partial_a E$, by

$$\partial_a : \mathrm{RegEx} \to \mathrm{RegEx}$$

$$\begin{aligned}
\partial_a \emptyset &= \emptyset, \\
\partial_a \varepsilon &= \emptyset, \\
\partial_a \boldsymbol{a} &= \varepsilon, \\
\partial_a \boldsymbol{b} &= \emptyset, \quad \text{for } a \neq b,
\end{aligned}$$

$$\begin{aligned}
\partial_a(E + F) &= \partial_a E + \partial_a F, \\
\partial_a(EF) &= \begin{cases} (\partial_a E)F + \partial_a F, & \text{if } \varepsilon \in \mathrm{L}(E), \\ (\partial_a E)F, & \text{otherwise,} \end{cases} \\
\partial_a(E^*) &= (\partial_a E)E^*.
\end{aligned}$$

Derivatives of Regular Expressions

Definition

Let E be a regular expression on Σ and let $w \in \Sigma^*$ be a string. We define recursively the **derivative** of E by w, denoted $\partial_w E$, by

$$\begin{aligned} \partial_w &: \mathrm{RegEx} \to \mathrm{RegEx} \\ \partial_\varepsilon E &= E, \\ \partial_{ax} E &= \partial_a(\partial_x E). \end{aligned}$$

Derivatives of Regular Expressions

Theorem (Brzozowski [1964], Theorem 4.2)

Let E be a regular expression on Σ and let $w \in \Sigma^*$ be a string. Then

$$w \in \mathrm{L}(E) \qquad \Leftrightarrow \qquad \varepsilon \in \mathrm{L}(\partial_w E).$$

Libraries

Observation

Theoretical regular expressions $\neq$ practical regular expressions.

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Some programming languages with support to regular expressions

.NET, C, Haskell, Java, Mathematica, MATLAB and Perl.

Algorithms

Algorithms

See the HASKELL implementation of some algorithms on regular expressions in the course homepage.

Applications

Some programs that use regular expressions

GREP:	Print lines matching a pattern
AWK:	Pattern scanning and processing language
SED:	Stream editor for filtering and transforming text
ALEX, FLEX and LEX:	Lexical-analyser generators
EMACS and VIM:	Test editors
MYSQL and ORACLE:	Databases

Applications

Reading

§ 3.3. Applications of Regular Expressions.

Applications

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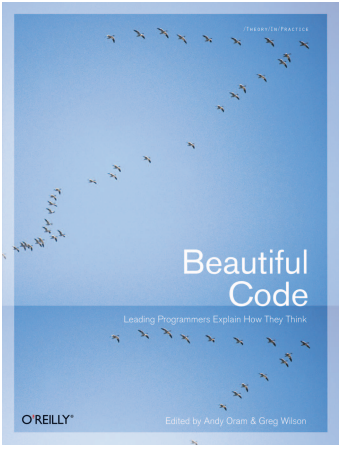
In the above section are defined:

$$E^+ := EE^* \qquad \text{(one or many times operator)}$$

$$E? := \varepsilon + E \qquad \text{(zero or one time operator)}$$

An Implementation: A Regular Expression Matcher



'Rob's implementation itself is a superb example of beautiful code: compact, elegant, efficient, and useful. It's one of the best examples of recursion that I have ever seen.'

Brian Kernighan, p. 3.

References

Brzozowski, J. A. (1964). Derivates of Regular Expressions. Journal of the ACM 11.4, pp. 481–494. DOI: 10.1145/321239.321249 (cit. on pp. 35–37, 40).

Hopcroft, J. E., Motwani, R. and Ullman, J. D. [1979] (2007). Introduction to Automata Theory, Languages, and Computation. 3rd ed. Pearson Education (cit. on p. 2).