CM0081 Formal Languages and Automata
§ 2.3 Non-Deterministic Finite Automata

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Preliminaries

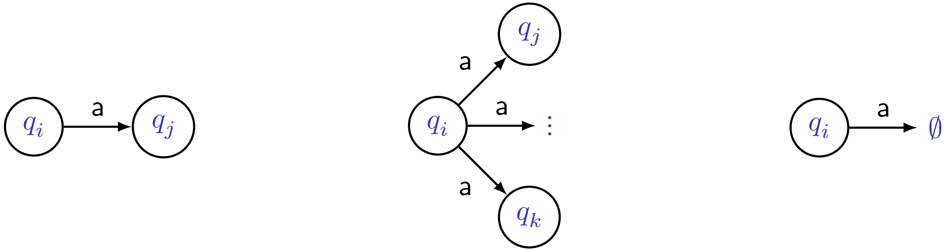
Conventions

- The number and page numbers assigned to chapters, examples, exercises, figures, quotes, sections and theorems on these slides correspond to the numbers assigned in the textbook [Hopcroft, Motwani and Ullman (1979) 2007].
- The natural numbers include the zero, that is, $\mathbb{N} = \{0, 1, 2, \dots\}$.
- The power set of a set A, that is, the set of its subsets, is denoted by $\mathcal{P}A$.

Non-Deterministic Finite Automata

Introduction

- The transition from a state and a symbol can be to: one state, various states or no state.



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Non-Deterministic Finite Automata

Introduction (continuation)

- Nondeterminism does not increase the computational power (or expressive power) of finite automata.

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Non-Deterministic Finite Automata

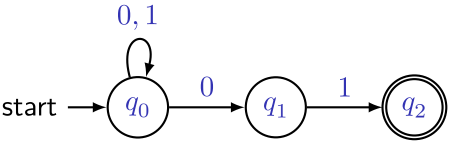
Introduction (continuation)

- Nondeterminism does not increase the computational power (or expressive power) of finite automata.
- The processing of an input by a non-deterministic finite automaton can be thought of in terms of guess and verify [Kozen (1997) 2012].
- Nondeterminism facilitates the design of the automata.

Non-Deterministic Finite Automata

Example

A non-deterministic finite automaton accepting all the binary strings that end in 01.



- q_0: The automaton 'guess' that the final 01 has not begun.
- q_1: The automaton 'guess' that the final 01 has begun.
- q_2: The word ends in 01.

Non-Deterministic Finite Automata

Definition

A **non-deterministic finite automaton** (NFA) is a 5-tuple

$$(Q, \Sigma, \delta, q_0, F),$$

where

(i) Q is the finite set of states,

(ii) Σ is the alphabet of input symbols,

(iii) $\delta : Q \times \Sigma \to \mathcal{P}Q$ is the transition function,

(iv) $q_0 \in Q$ is the start state,

(v) $F \subseteq Q$ is the set of accepting (or final) states.

Extension of the Transition Function for NFAs

Definition

Let $N = (Q, \Sigma, \delta, q_0, F)$ be a NFA. The **extension of the transition function**, denoted by $\hat{\delta}$, is recursively defined by

$$\hat{\delta} : Q \times \Sigma^* \to \mathcal{P}Q$$

$$\hat{\delta}(q, \varepsilon) = \{q\},$$

$$\hat{\delta}(q, xa) = \bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a).$$

Languages Accepted by NFAs

Recall

Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Recall that the language accepted by D was defined by

$$\mathrm{L}(D) := \left\{ w \in \Sigma^* \;\middle|\; \hat{\delta}(q_0, w) \in F \right\}.$$

Languages Accepted by NFAs

Definitions

Let $N = (Q, \Sigma, \delta, q_0, F)$ be a NFA and let $w \in \Sigma^*$ be a string.

(i) The string w is **accepted** by N iff $\hat{\delta}(q_0, w) \cap F \neq \emptyset$.

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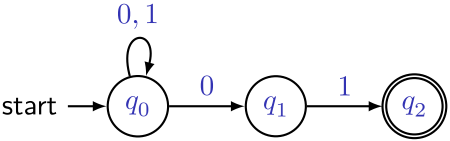
Reading

§ 2.4. An application: Text search.

Languages Accepted by NFAs

Example 2.9

For the NFA of the figure, $\mathrm{L}(N) = \{\, w \in \{0, 1\}^* \mid w$ ends in $01 \,\}$.



Languages Accepted by NFAs

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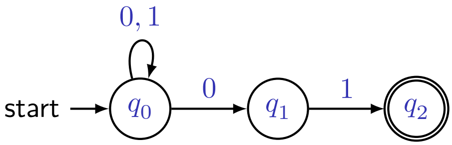

Sketch of proof

Mutual induction on the following propositions:

$S_0(w)$: $q_0 \in \hat{\delta}(q_0, w)$ for all $w \in \Sigma^*$
$S_1(w)$: $q_1 \in \hat{\delta}(q_0, w) \Leftrightarrow w$ ends in 0
$S_2(w)$: $q_2 \in \hat{\delta}(q_0, w) \Leftrightarrow w$ ends in 01

From $S_2(w)$ and $F = \{q_2\}$ the theorem follows.

Languages Accepted by NFAs

Example (Exercise 2.3.4.a)

NFA accepting the set of strings over $\Sigma = \{1, 2, 3\}$ such that the final digit has appeared before.

Languages Accepted by NFAs

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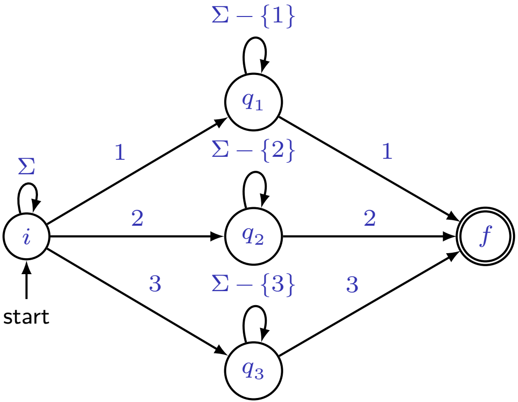

- q_i: The automaton 'guess' that the repeated digit is i.

Languages Accepted by NFAs

Example (Exercise 2.3.4.b)

NFA accepting the set of strings over $\Sigma = \{0, 1, 2\}$ such that the final digit has not appeared before.

Languages Accepted by NFAs

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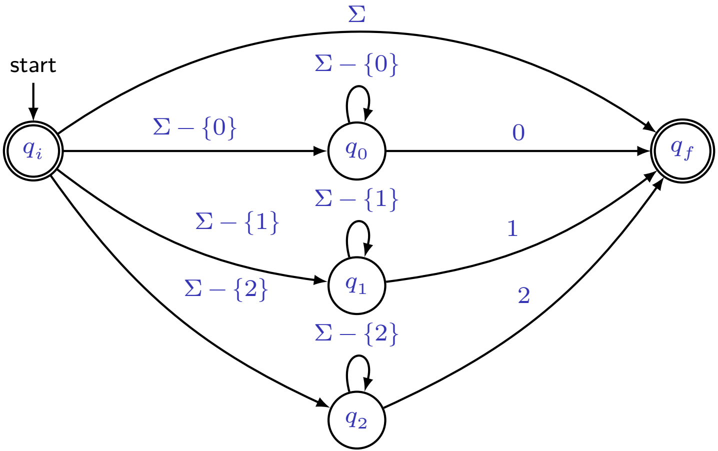

Languages Accepted by NFAs

Example (Exercise 2.3.4.c)

NFA accepting the set of strings over $\Sigma = \{0, 1\}$ such that there are two 0's separated by a number of positions that is multiple of 2. Note that 0 is an allowable multiple of 2.

Languages Accepted by NFAs

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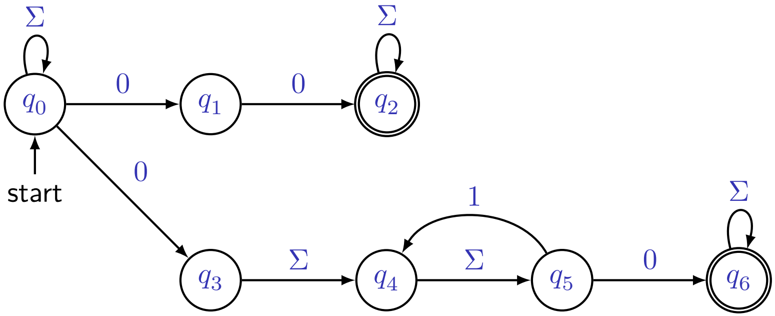

Languages Accepted by NFAs

Example (Exercise 2.5.3.b)

NFA accepting the set of strings that consist of either 01 repeated one or more times or 010 repeated one or more times.

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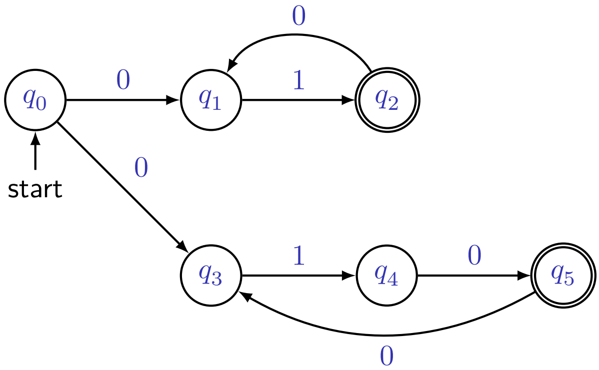

The Subset Construction

Construction

Input: A NFA $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$
Output: A DFA $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$ where

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$$Q_D = \mathcal{P}Q_N,$$

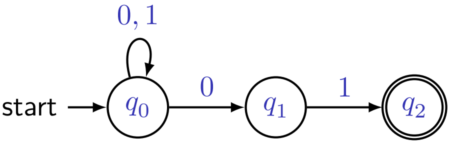
$$F_D = \{\, S \in \mathcal{P}Q_N \mid S \cap F_N \neq \emptyset \,\},$$

$$\delta_D(S, a) = \bigcup_{p \in S} \delta_N(p, a), \text{ for each } S \in \mathcal{P}Q_N \text{ and } a \in \Sigma.$$

The Subset Construction

Example

Given the following NFA to build the DFA given by the subset construction.



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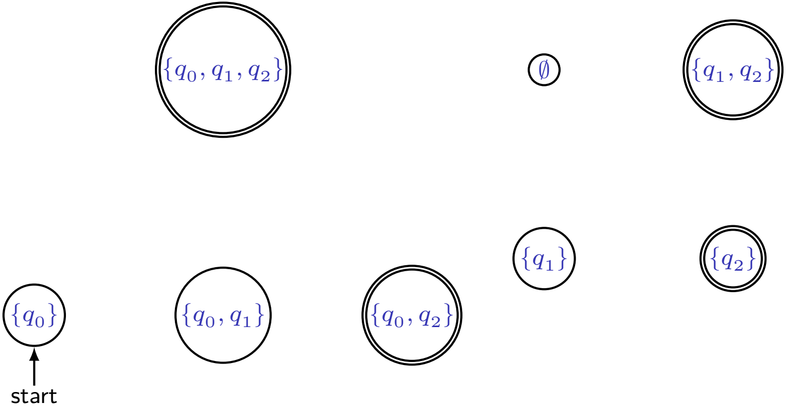
The Subset Construction

Example (continuation)



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The Subset Construction

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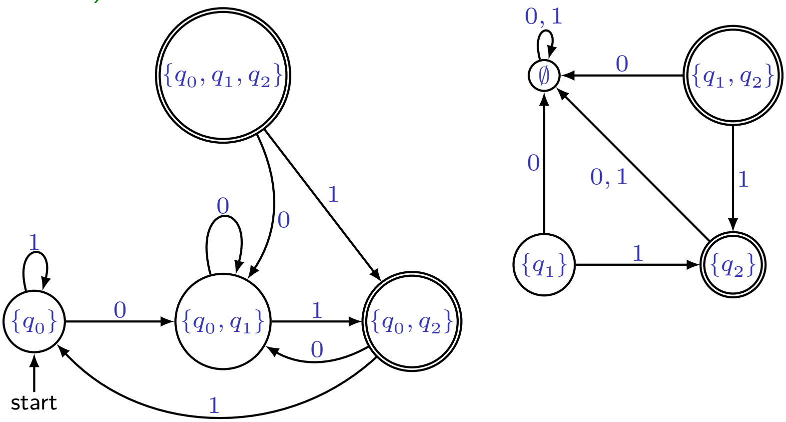

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The Subset Construction

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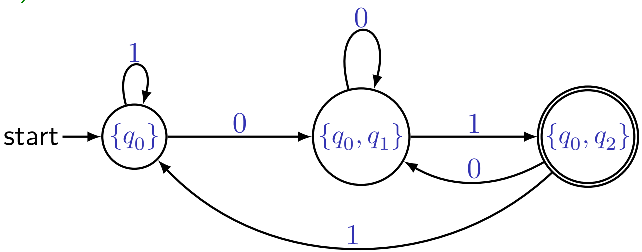
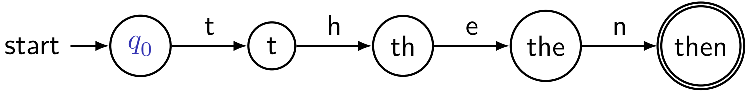


The Subset Construction

Example (special case)

A NFA recognising the word ‘then’:



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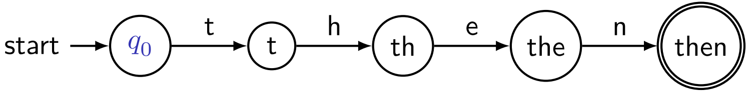
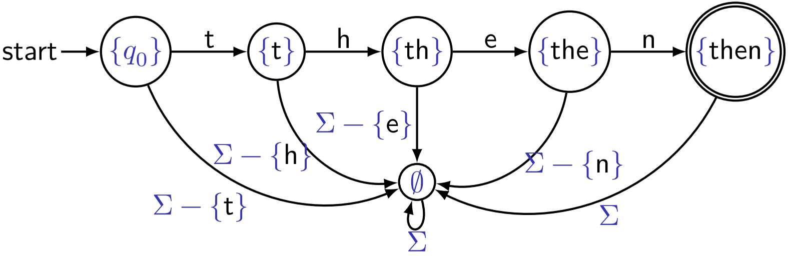


The DFA given by the subset construction:



Equivalence of DFAs and NFAs

Theorem 2.11

Let $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$ be the DFA constructed from a NFA $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$ by the subset construction. Then

$$\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$$

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Proof (by structural induction on w)

In the blackboard.

Equivalence of DFAs and NFAs

Theorem 2.12

A language L is accepted by some DFA iff L is accepted by some NFA.

Equivalence of DFAs and NFAs

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Definition

A language L is **regular** iff exists a finite automaton A (DFA or NFA) such that $L = \mathrm{L}(A)$.

References

- Hopcroft, J. E., Motwani, R. and Ullman, J. D. [1979] (2007). Introduction to Automata Theory, Languages, and Computation. 3rd ed. Pearson Education (cit. on p. 2).
- Kozen, D. C. [1997] (2012). Automata and Computability. Third printing. Undergraduate Texts in Computer Science. Springer. DOI: 10.1007/978-1-4612-1844-9 (cit. on pp. 4–6).