CM0246 Discrete Structures
Boolean Algebras

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, sections, and theorems on these slides correspond to the numbers assigned in the textbook (Rosen 2004).

Boolean Functions

Boolean operations

We define the following operations in the set $B = \{0, 1\}$:

- Boolean sum

  $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$ and $1 + 1 = 1$.

- Boolean product

  $0 \cdot 0 = 0$, $0 \cdot 1 = 0$, $1 \cdot 0 = 0$ and $1 \cdot 1 = 1$.

- Complement

  $\overline{0} = 1$ and $\overline{1} = 0$.

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Example

Whiteboard.

Boolean Functions

From Boolean operations/logical operators to logical operators/Boolean operations

Boolean operations	logic operators
$\cdot$	$\wedge$
$+$	$\vee$
$^{-}$	$\neg$
0	F
1	T

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Example (from equality/logical equivalence to logical equivalence/equality)

Whiteboard.

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Definition

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Example

Whiteboard.

Boolean Functions

Theorem (Example 5, p. 280)

If $|A| = m$ and $|B| = n$ then $|\{f : A \to B\}| = n^m$.

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Example

There are 16 Boolean functions of degree 2.

Boolean Functions

Definition

Let $x_1, x_2, \ldots, x_n$ be Boolean variables. The **Boolean expressions** are inductively defined by

- Basis step: 0, 1 and $x_1, x_2, \ldots, x_n$ are Boolean expressions.
- Inductive step: If E_1 and E_2 are Boolean expressions then $\overline{E_1}$, $(E_1 \cdot E_2)$ and $(E_1 + E_2)$ are Boolean expressions.

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Each Boolean expression represents a Boolean function.

Example

Whiteboard.

Logical Equivalences

Identity laws

$p \wedge \mathrm{T} \equiv p$

$p \vee \mathrm{F} \equiv p$

Domination laws

$p \wedge \mathrm{F} \equiv \mathrm{F}$

$p \vee \mathrm{T} \equiv \mathrm{T}$

Idempotent laws

$p \wedge p \equiv p$

$p \vee p \equiv p$

Double negation law

$\neg(\neg p) \equiv p$

Commutative laws

$p \wedge q \equiv q \wedge p$

$p \vee q \equiv q \vee p$

Associate laws

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

Distributive laws

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

De Morgan's laws

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Absorption laws

$p \wedge (p \vee q) \equiv p$

$p \vee (p \wedge q) \equiv p$

Negation laws

$p \wedge \neg p \equiv \mathrm{F}$

$p \vee \neg p \equiv \mathrm{T}$

Boolean Identities

Identity laws

$x \cdot 1 = x$

$x + 0 = x$

Domination laws

$x \cdot 0 = 0$

$x + 1 = 1$

Idempotent laws

$x \cdot x = x$

$x + x = x$

Double complement law

$\overline{\overline{x}} = x$

Commutative laws

$x \cdot y = y \cdot x$

$x + y = y + x$

Associate laws

$(x \cdot y) \cdot z = x \cdot (y \cdot z)$

$(x + y) + z = x + (y + z)$

Distributive laws

$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

$x + (y \cdot z) = (x + y) \cdot (x + z)$

De Morgan's laws

$\overline{x \cdot y} = \overline{x} + \overline{y}$

$\overline{(x + y)} = \overline{x} \cdot \overline{y}$

Absorption laws

$x \cdot (x + y) = x$

$x + x \cdot y = x$

Complement laws

$x \cdot \overline{x} = 0$

$x + \overline{x} = 1$

Boolean Identities

Each Boolean identity can be proved using a table.

Example

Whiteboard.

Boolean Algebras

Definition

Let $\wedge$ and $\vee$ be two binaries operations, $^-$ a unary operation and 0 and 1 two constants. A **Boolean algebra** is an algebraic structure $(B, \wedge, \vee, ^-, 0, 1)$, which satisfy the following axioms for all x, y and z in B:

Identity laws

$x \wedge 1 = x$

$x \vee 0 = x$

Complement laws

$x \wedge \overline{x} = 0$

$x \vee \overline{x} = 1$

Associate laws

$(x \wedge y) \wedge z = x \wedge (y \wedge z)$

$(x \vee y) \vee z = x \vee (y \vee z)$

Commutative laws

$x \wedge y = y \wedge x$

$x \vee y = y \vee x$

Distributive laws

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Boolean Algebra

Example

$(B, \wedge, \vee, ^-, 0, 1)$	Set theory	Propositional logic	$(\{0,1\}, \cdot, +, ^-, 0, 1)$
B	U	set of formulae	$\{0,1\}$
$\wedge$	$\cap$	$\wedge$	$\cdot$
$\vee$	$\cup$	$\vee$	$+$
$^-$	$^-$	$\neg$	$^-$
0	$\emptyset$	F	0
1	U	T	1

Boolean Algebra

Definition

The **dual** of any statement in a Boolean algebra $(B, \wedge, \vee, ^{-}, 0, 1)$ is the statement obtained by interchanging $\wedge$ and $\vee$, and interchanging 0 and 1.

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Example

The dual of $x \wedge (y \vee 0)$ is $x \vee (y \wedge 1)$.

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Theorem (Principle of duality. Problem 38, p. 660)

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Theorem (Principle of duality. Problem 38, p. 660)

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Proof.

Similar to the proof of the principle of duality for lattices. ■

Boolean Algebra

Problem 31 (p. 660)

Let $\mathbb{B} = (B, \wedge, \vee, ^{-}, 0, 1)$ be a Boolean algebra. To prove that $\mathbb{B}$ satisfy the idempotent laws $x \vee x = x$ and $x \wedge x = x$, for every element x.

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Proof.

$$
\begin{aligned}
x \vee x &= (x \vee x) \wedge 1 && \text{(identity law)} \\
&= (x \vee x) \wedge (x \vee \overline{x}) && \text{(complement law)} \\
&= x \vee (x \wedge \overline{x}) && \text{(distributive law)} \\
&= x \vee 0 && \text{(complement law)} \\
&= x && \text{(identity law)}
\end{aligned}
$$

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Proof.

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x \vee x &= (x \vee x) \wedge 1 && \text{(identity law)} & x \wedge x &= (x \wedge x) \vee 0 \\
&= (x \vee x) \wedge (x \vee \overline{x}) && \text{(complement law)} & &= (x \wedge x) \vee (x \wedge \overline{x}) \\
&= x \vee (x \wedge \overline{x}) && \text{(distributive law)} & &= x \wedge (x \vee \overline{x}) \\
&= x \vee 0 && \text{(complement law)} & &= x \wedge 1 \\
&= x && \text{(identity law)} & &= x
\end{aligned}$$

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&= x \vee 0 && \text{(complement law)} & &= x \wedge 1 \\
&= x && \text{(identity law)} & &= x
\end{aligned}
$$

■

Remark: $x \wedge x = x / x \vee x = x$ also follows from $x \vee x = x / x \wedge x = x$ by the principle of duality.

Boolean Algebra

Problem 34 (p. 660)

Let $\mathbb{B} = (B, \wedge, \vee, ^-, 0, 1)$ be a Boolean algebra. To prove that $\mathbb{B}$ satisfy the double complement law, i.e. $\forall x(x = \overline{\overline{x}})$.

Boolean Algebra

Problem 34 (p. 660)

Let $\mathbb{B} = (B, \wedge, \vee, ^{-}, 0, 1)$ be a Boolean algebra. To prove that $\mathbb{B}$ satisfy the double complement law, i.e. $\forall x(x = \overline{\overline{x}})$.

Hint: From the complement laws, we have

$$\overline{x} \wedge \overline{\overline{x}} = 0,$$
$$\overline{x} \vee \overline{\overline{x}} = 1.$$

Proved on next slide

Boolean Algebra

Proof (Lipschutz 1994).

$$
\begin{aligned}
x &= x \vee 0 && \text{(identity law)}\\
&= x \vee (\overline{x} \wedge \overline{\overline{x}}) && \text{(complement law)}\\
&= (x \vee \overline{x}) \wedge (x \vee \overline{\overline{x}}) && \text{(distributive law)}\\
&= 1 \wedge (x \vee \overline{\overline{x}}) && \text{(complement law)}\\
&= (\overline{x} \vee \overline{\overline{x}}) \wedge (x \vee \overline{\overline{x}}) && \text{(complement law)}\\
&= (\overline{\overline{x}} \vee \overline{x}) \wedge (\overline{\overline{x}} \vee x) && \text{(commutative law)}\\
&= \overline{\overline{x}} \vee (\overline{x}) \wedge x) && \text{(distributive law)}\\
&= \overline{\overline{x}} \vee (x \wedge \overline{x}) && \text{(commutative law)}\\
&= \overline{\overline{x}} \vee 0 && \text{(complement law)}\\
&= \overline{\overline{x}} && \text{(identity law)} \quad \blacksquare
\end{aligned}
$$

References

- Lipschutz, S. (1994). Teoría de Conjuntos y Temas Afines. Serie Schaum. McGraw-Hill (cit. on p. 29).
- Rosen, K. H. (2004). Matemática Discreta y sus Aplicaciones. Trans. by Pérez Morales, J. M., Moro Carreño, J., Lías Quintero, A. I. and Ramos Alonc, P. A. 5th ed. McGraw-Hill (cit. on p. 2).