Category Theory and Functional Programming
Universality and Adjoints

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Semester 2022-2

Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, pages, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Abramsky and Tzevelekos 2011].

Outline

Adjunctions for Pre-orders

Definition

Let $(P, \preceq_P)$ and $(Q, \preceq_Q)$ be two pre-ordered sets and let f and g be two homomorphisms (monotone maps) forth and back.

$$\begin{tikzcd} P \arrow[r, bend left, "f"] & Q \arrow[l, bend left, "g"] \end{tikzcd}$$

The maps f and g are **adjoint**, denoted $f \dashv g$, when for all $x \in P$, $y \in Q$,

$$f\,x \preceq_Q y \quad \text{iff} \quad x \preceq_P g\,y.$$

The map f is the **left adjoint** and the map g is the **right adjoint** [Awodey and Bauer 2022].

Universal Arrows

Universal Arrows

Definition

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor and let C be an object of $\mathcal{C}$. A **universal arrow** from C to G is a pair (D, η) where D is an object of $\mathcal{D}$ and η is an arrow

$$\eta : C \to G_0\, D$$

Exponentials

Exponentials

Introduction

We shall introduce abstract characterisations of the following facts of set theory:

(i) The functions of a set X to a set Y, denoted Y^X, are also a set.

(ii) A function of two variables $f : X \times Y \to Z$ can be represented as a function of one variable $\lambda f : X \to Z^Y$.

Exponentials

Set theory

Let Z^Y be the set of functions from the set Y to the set Z. For any set X and function $g : X \times Y \to Z$, the following diagrams commutes:

(continued on next slide)

Exponentials

Set theory (continuation)

$$
\begin{tikzcd}
Z^Y \\
X \arrow[u, "\Lambda g"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Z^Y \times Y \arrow[r, "\mathrm{eval}_{Y,Z}"] & Z \\
X \times Y \arrow[u, "\Lambda g \times \mathrm{id}_Y"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\Big(g = \mathrm{eval}_{Y,Z} \circ (\Lambda g \times \mathrm{id}_Y)\Big)
$$

where

$$\mathrm{eval}_{Y,Z} : (Z^Y \times Y) \to Z := (f, y) \mapsto f\, y \qquad \text{(application)},$$
$$\Lambda g : X \to (Y \to Z) := x\, y \mapsto g(x, y) \qquad \text{(currying)}.$$

Exponentials

Definition

Let $\mathcal{C}$ be a category with binary products, and let Z and Y be objects of $\mathcal{C}$. The pair (Z^Y, eval), where Z^Y is an object of $\mathcal{C}$ and eval is an arrow

$$\mathrm{eval} : (Z^Y \times Y) \to Z$$

is an **exponential object** iff for any object X and arrow $g : X \times Y \to Z$, there is a unique arrow $\Lambda g : X \to Z^Y$ such that the following diagram commutes:

(continued on next slide)

Exponentials

Definition (continuation)

$$
\begin{tikzcd}
Z^Y & Z^Y \times Y \arrow[r, "\mathrm{eval}_{Y,Z}"] & Z \\
X \arrow[u, dotted, "\Lambda g"] & X \times Y \arrow[u, dotted, "\Lambda g \times \mathrm{id}_Y"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

$$\Big(g = \mathrm{eval}_{Y,Z} \circ (\Lambda g \times \mathrm{id}_Y)\Big)$$

Cartesian Closed Categories

Cartesian Closed Categories

Definition

A category is a **Cartesian closed category** iff it has finite products (i.e. a terminal object and binary products) and exponentials.

References

References

Abramsky, S. and Tzevelekos, N. (2011). Introduction to Categories and Categorical Logic. In: New Structures for Physics. Ed. by Coecke, B. Vol. 813. Lecture Notes in Physics. Springer, pp. 3–94. DOI: 10.1007/978-3-642-12821-9_1 (cit. on p. 2).

Awodey, S. and Bauer, A. (2022). Introduction to Categorical Logic. Appendix A. Category Theory. Draft version 2022-01-15. URL: https://awodey.github.io/catlog/ (cit. on p. 4).