

Category Theory and Functional Programming

Universality and Adjoint

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, pages, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Abramsky and Tzevelekos 2011].

Outline

Adjunctions for Pre-orders

Universal Arrows

Exponentials

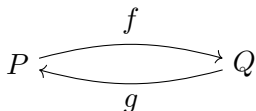
Cartesian Closed Categories

References

Adjunctions for Pre-orders

Definition

Let $(P, \preceq_P)$ and $(Q, \preceq_Q)$ be two pre-ordered sets and let f and g be two homomorphisms (monotone maps) forth and back.



The maps f and g are **adjoint**, denoted $f \dashv g$, when for all $x \in P$, $y \in Q$,

$$f x \preceq_Q y \quad \text{iff} \quad x \preceq_P g y.$$

The map f is the **left adjoint** and the map g is the **right adjoint** [Awodey and Bauer 2022].

Universal Arrows

Universal Arrows

Definition

Let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a functor and let C be an object of $\mathcal{C}$. A **universal arrow** from C to G is a pair (D, η) where D is an object of $\mathcal{D}$ and η is an arrow

$$\eta : C \rightarrow G_0 D$$

Exponentials

Exponentials

Introduction

We shall introduce abstract characterisations of the following facts of set theory:

- (i) The functions of a **set** X to a **set** Y , denoted Y^X , are also a **set**.
- (ii) A function of **two** variables $f : X \times Y \rightarrow Z$ can be represented as a function of **one** variable $\lambda f : X \rightarrow Z^Y$.

Exponentials

Set theory

Let Z^Y be the set of functions from the set Y to the set Z . For any set X and function $g : X \times Y \rightarrow Z$, the following diagrams commutes:

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Exponentials

Set theory (continuation)

$$\begin{array}{ccc} & & Z^Y \times Y \xrightarrow{\text{eval}_{Y,Z}} Z \\ & \nearrow & \\ & \Lambda g \times \text{id}_Y & \\ & \uparrow & \\ X & \Lambda g & Z^Y \end{array}$$

$$(g = \text{eval}_{Y,Z} \circ (\Lambda g \times \text{id}_Y))$$

where

$$\text{eval}_{Y,Z} : (Z^Y \times Y) \rightarrow Z := (f, y) \mapsto f y$$

(application),

$$\Lambda g : X \rightarrow (Y \rightarrow Z) := x y \mapsto g(x, y)$$

(currying).

Exponentials

Definition

Let $\mathcal{C}$ be a category with binary products, and let Z and Y be objects of $\mathcal{C}$. The pair (Z^Y, eval) , where Z^Y is an object of $\mathcal{C}$ and eval is an arrow

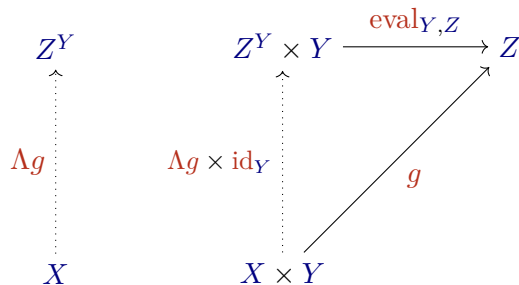
$$\text{eval} : (Z^Y \times Y) \rightarrow Z$$

is an **exponential object** iff for any object X and arrow $g : X \times Y \rightarrow Z$, there is a **unique** arrow $\Lambda g : X \rightarrow Z^Y$ such that the following diagram commutes:

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Exponentials

Definition (continuation)



$$(g = \text{eval}_{Y,Z} \circ (\Lambda g \times \text{id}_Y))$$

Cartesian Closed Categories

Cartesian Closed Categories

Definition

A category is a **Cartesian closed category** iff it has finite products (i.e. a terminal object and binary products) and exponentials.

References

References



Abramsky, S. and Tzevelekos, N. (2011). Introduction to Categories and Categorical Logic. In: New Structures for Physics. Ed. by Coecke, B. Vol. 813. Lecture Notes in Physics. Springer, pp. 3–94. DOI: [10.1007/978-3-642-12821-9_1](https://doi.org/10.1007/978-3-642-12821-9_1) (cit. on p. 2).



Awodey, S. and Bauer, A. (2022). Introduction to Categorical Logic. Appendix A. Category Theory. Draft version 2022-01-15. URL: <https://awodey.github.io/catlog/> (cit. on p. 4).