Category Theory and Functional Programming
Natural Transformations

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, pages, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Abramsky and Tzevelekos 2011].

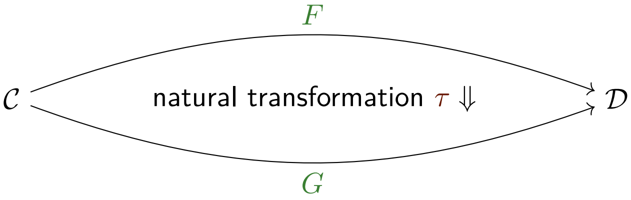
Outline

Introduction

Introduction

Description

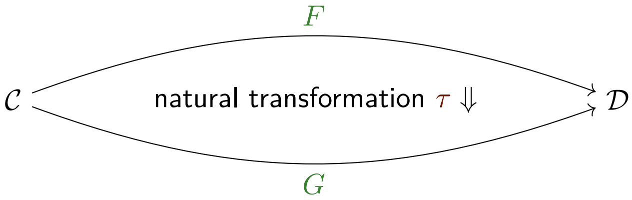
A natural transformation is a structure preserving mapping (i.e. preserves composition of arrows and identity arrows) between 'parallel' functors.



Introduction

Description

A natural transformation is a structure preserving mapping (i.e. preserves composition of arrows and identity arrows) between 'parallel' functors.



'As Eilenberg-Mac Lane first observed, "category" has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformation".' [Mac Lane 1998, p. 48]

Definition of a Natural Transformation

Definition of a Natural Transformation

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation**[†]

$$\tau : F \Rightarrow G$$

is a family of arrows in $\mathcal{D}$ indexed by objects A of $\mathcal{C}$,

$$\{\, \tau_A : F_0\, A \to G_0\, A \,\}_{A \in \mathrm{Obj}(\mathcal{C})} \qquad \textbf{(components of } \tau \textbf{ at } A\textbf{)}$$

such that, for all $A \xrightarrow{f} B$ in $\mathcal{C}$,

$$(G_1\, f) \circ \tau_A = \tau_B \circ (F_1\, f) \qquad \textbf{(naturality condition)}.$$

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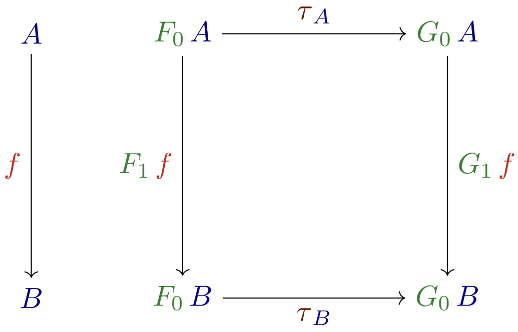
[†] The textbook uses the notation $\tau : F \to G$.

Definition (continuation)

That is, the following diagram commutes.



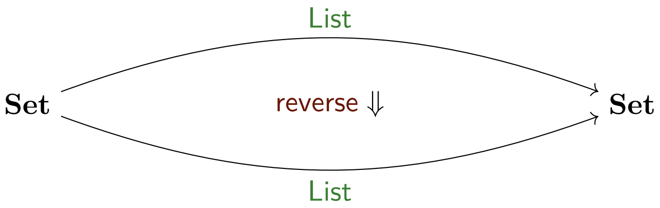
$$\big((G_1\ f) \circ \tau_A = \tau_B \circ (F_1\ f)\big)$$

Examples of Natural Transformations

Examples of Natural Transformations

Example

We shall define the natural transformation reverse on the functor List : $\mathbf{Set} \to \mathbf{Set}$.



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Examples of Natural Transformations

Example (continuation)

$$\text{reverse} : \text{List} \Rightarrow \text{List}$$

$$\text{reverse}_X : \text{List}_0\, X \to \text{List}_0\, X$$

$$\text{reverse}_X : [X] \to [X]$$

$$\text{reverse}_X\, [x_1, \ldots, x_n] := [x_n, \ldots, x_1]$$

Examples of Natural Transformations

Example (continuation)

$\mathsf{reverse} : \mathsf{List} \Rightarrow \mathsf{List}$

$\mathsf{reverse}_X : \mathsf{List}_0\, X \to \mathsf{List}_0\, X$

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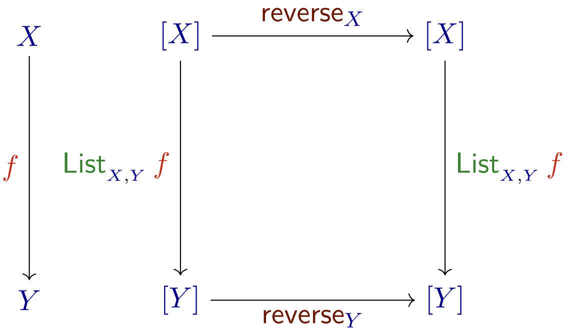
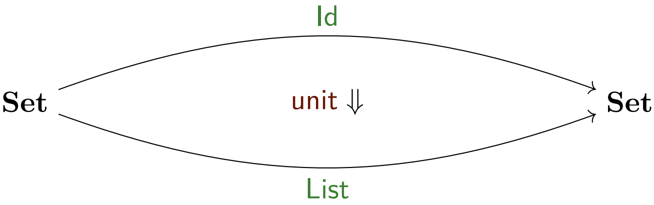


For each $f : X \to Y$ in $\mathbf{Set}$, the above diagram commutes by naturality.

Examples of Natural Transformations

Example

We shall define the natural transformation unit on the functors $\mathsf{Id}, \mathsf{List} : \mathbf{Set} \to \mathbf{Set}$.



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Examples of Natural Transformations

Example (continuation)

$$\mathsf{unit} : \mathsf{Id} \Rightarrow \mathsf{List}$$

$$\mathsf{unit}_X : \mathsf{Id}_0\, X \to \mathsf{List}_0\, X$$

$$\mathsf{unit}_X : X \to [X]$$

$$\mathsf{unit}_X\, x := [x]$$

Examples of Natural Transformations

Example (continuation)

$$\mathsf{unit} : \mathsf{Id} \Rightarrow \mathsf{List}$$
$$\mathsf{unit}_X : \mathsf{Id}_0\, X \to \mathsf{List}_0\, X$$
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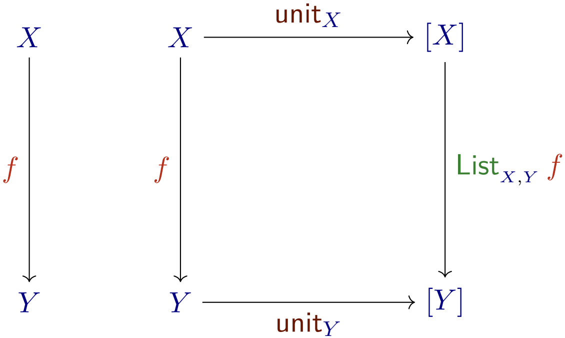
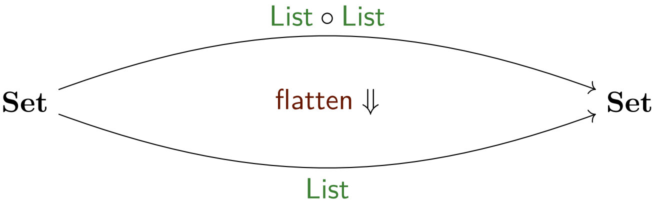


For each $f : X \to Y$ in $\mathbf{Set}$, the above diagram commutes by naturality.

Examples of Natural Transformations

Example

We define the natural transformation flatten on the functors $\mathsf{List} \circ \mathsf{List}, \mathsf{List} : \mathbf{Set} \to \mathbf{Set}$.



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Examples of Natural Transformations

Example (continuation)

$$\mathsf{flatten} : \mathsf{List} \circ \mathsf{List} \Rightarrow \mathsf{List}$$

$$\mathsf{flatten}_X : \mathsf{List}_0\,(\mathsf{List}_0\, X) \to \mathsf{List}_0\, X$$

$$\mathsf{flatten}_X : [[X]] \to [X]$$

$$\mathsf{flatten}_X\, [[x^1_1, \ldots, x^1_{n_1}], \ldots, [x^k_1, \ldots, x^k_{n_k}]] := [x^1_1, \ldots, x^1_{n_1}, \ldots, x^k_1, \ldots, x^k_{n_k}]$$

Examples of Natural Transformations

Example (continuation)

$$\mathsf{flatten} : \mathsf{List} \circ \mathsf{List} \Rightarrow \mathsf{List}$$

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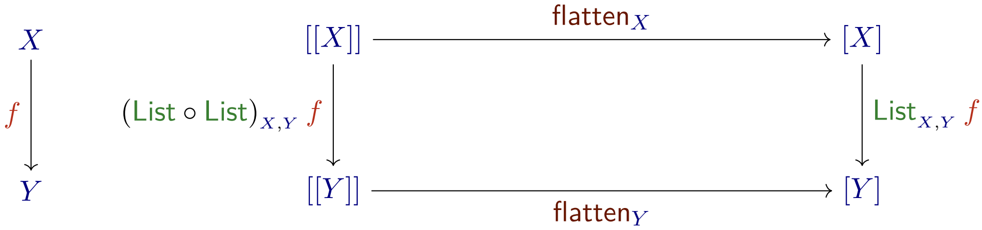
For each $f : X \to Y$ in $\mathbf{Set}$, the following diagram commutes by naturality.



Examples of Natural Transformations

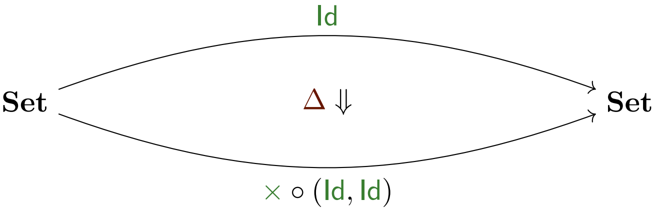
Exercise 1

Verify naturality of reverse, unit and flatten.

Examples of Natural Transformations

Example

Let Id be the identity functor in $\mathbf{Set}$ and let $\times \circ (\mathsf{Id}, \mathsf{Id}) : \mathbf{Set} \to \mathbf{Set}$ the functor sending every set X to $X \times X$ and every function f to $f \times f$. We shall define the natural transformation **diagonal** Δ.



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Examples of Natural Transformations

Example (continuation)

$$\Delta : \mathsf{Id} \Rightarrow \times \circ (\mathsf{Id}, \mathsf{Id})$$
$$\Delta_X : \mathsf{Id}_0\, X \to (\times \circ (\mathsf{Id}, \mathsf{Id}))_0\, X$$
$$\Delta_X : X \to X \times X$$
$$\Delta_X\, x := (x, x)$$

Examples of Natural Transformations

Example (continuation)

$$\Delta : \mathsf{Id} \Rightarrow \times \circ (\mathsf{Id}, \mathsf{Id})$$
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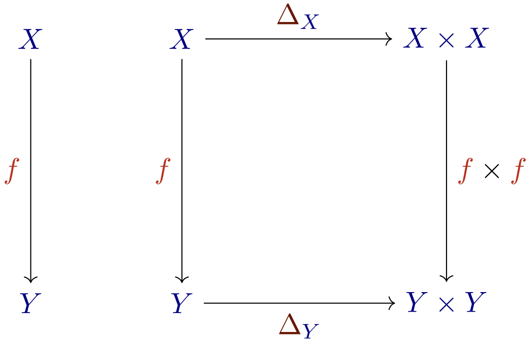
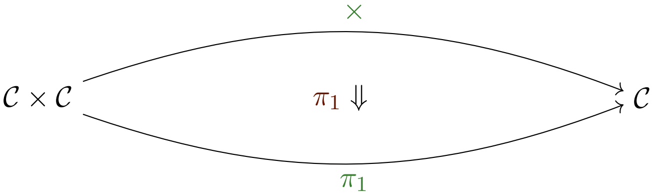


For each $f : X \to Y$ in $\mathbf{Set}$, the above diagram commutes by naturality.

Examples of Natural Transformations

Example

Let $\times, \pi_1 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be the product and first projection functors, respectively. We shall define the natural transformation **first projection** π_1.



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Examples of Natural Transformations

Example (continuation)

$$\pi_1 : \times \Rightarrow \pi_1$$

$$\pi_{1(A,B)} : \times_0 (A, B) \to (\pi_1)_0 (A, B)$$

$$\pi_{1(A,B)} : A \times B \to A$$

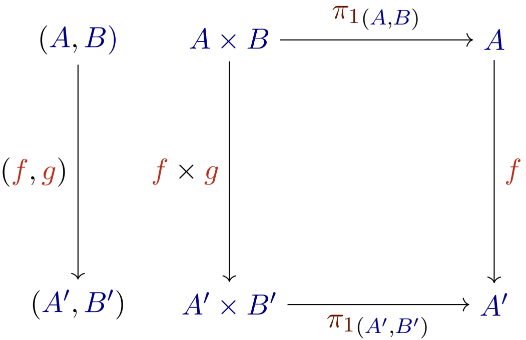
Examples of Natural Transformations

Example (continuation)

$\pi_1 : \times \Rightarrow \pi_1$

$\pi_{1(A,B)} : \times_0 (A, B) \to (\pi_1)_0 (A, B)$

$\pi_{1(A,B)} : A \times B \to A$

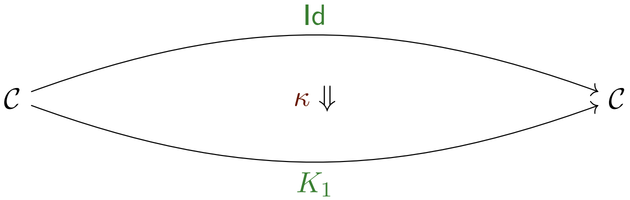


For each $(f, g) : (A, B) \to (A', B')$ in $\mathcal{C} \times \mathcal{C}$, the above diagram commutes by naturality.

Examples of Natural Transformations

Example

Let $\mathcal{C}$ be a category with terminal object 1, let Id be the identity functor in $\mathcal{C}$ and let $K_1 : \mathcal{C} \to \mathcal{C}$ be the functor mapping all objects of $\mathcal{C}$ to 1 and all arrows of $\mathcal{C}$ to id_1. We shall define the natural transformation κ.



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Examples of Natural Transformations

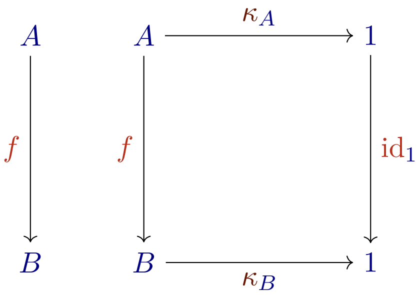
Example (continuation)

$$\begin{aligned} \kappa &: \mathsf{Id} \Rightarrow K_1 \\ \kappa_A &: \mathsf{Id}_0\, A \to (K_1)_0\, A \\ \kappa_A &: A \to 1 \end{aligned}$$

Examples of Natural Transformations

Example (continuation)

$$\kappa : \mathsf{Id} \Rightarrow K_1$$
$$\kappa_A : \mathsf{Id}_0\, A \to (K_1)_0\, A$$
$$\kappa_A : A \to 1$$



For each $f : A \to B$ in $\mathcal{C}$, the above diagram commutes by naturality.

Exercises

Exercise 2

Verify naturality of the natural transformation κ.

Natural Isomorphisms

Natural Isomorphisms

Definition

A natural transformation

$$\tau : F \Rightarrow G$$
$$\tau_A : F_0\, A \to G_0\, A, \quad \text{for all } A \text{ in } \mathcal{C}$$

is a **natural isomorphism** iff each τ_A is an isomorphism.

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Notation

A natural isomorphism is denoted by $\tau : F \overset{\cong}{\Longrightarrow} G$.

Natural Isomorphisms

Example

Let $\mathcal{C}$ be a category with products and let the functors (textbook and [Awodey 2010, § 7.4])

$$-_1 \times (-_2 \times -_3) : \mathcal{C}^3 \to \mathcal{C} \qquad \text{and} \qquad (-_1 \times -_2) \times -_3 : \mathcal{C}^3 \to \mathcal{C}.$$

The natural isomorphism a shows that the product is associative.

$$a : -_1 \times (-_2 \times -_3) \overset{\cong}{\Longrightarrow} (-_1 \times -_2) \times -_3$$

$$a_{A,B,C} : A \times (B \times C) \xrightarrow{\cong} (A \times B) \times C$$

$$a_{A,B,C} := \langle \langle \pi_1, \pi_1 \circ \pi_2 \rangle, \pi_2 \circ \pi_2 \rangle.$$

Example

Let $\mathcal{C}$ be a category with binary products. We define the functor $\bar{\times} : \mathcal{C}^2 \to \mathcal{C}$ by (textbook and [Awodey 2010, Example 7.8])

$$\bar{\times}_0\,(A, B) := \times_0\,(B, A) = B \times A, \qquad \bar{\times}_1\,(f, g) := \times_1\,(g, f) = g \times f.$$

The natural isomorphism s shows that the product is symmetric.

$$s : \times \overset{\cong}{\Longrightarrow} \bar{\times}$$

$$s_{A,B} : A \times B \xrightarrow{\cong} B \times A$$

$$s_{A,B} := \langle \pi_2, \pi_1 \rangle.$$

$$\begin{array}{c} (A, B) \\ {\scriptstyle (f,g)}\downarrow \\ (A', B') \end{array} \qquad \begin{array}{ccc} A \times B & \xrightarrow{s_{A,B}} & B \times A \\ {\scriptstyle f \times g}\downarrow & & \downarrow{\scriptstyle g \times f} \\ A' \times B' & \xrightarrow[s_{A',B'}]{} & B' \times A' \end{array}$$

Natural Isomorphisms

Example

Let $\mathcal{C}$ be a category with binary products and terminal object 1 and let the functors

$$\begin{aligned} &1 \times - : \mathcal{C} \to \mathcal{C} \\ &(1 \times -)_0\, A := 1 \times A \\ &(1 \times -)_1\, f := \langle \mathrm{id}_1, f \rangle, \end{aligned}$$

$$\begin{aligned} &- \times 1 : \mathcal{C} \to \mathcal{C} \\ &(- \times 1)_0\, A := A \times 1 \\ &(- \times 1)_1\, f := \langle f, \mathrm{id}_1 \rangle. \end{aligned}$$

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Natural Isomorphisms

Example (continuation)

The natural isomorphisms l and r show that 1 is the unit of the product.

$$
\begin{aligned}
l &: 1 \times - \overset{\cong}{\Longrightarrow} \mathsf{Id} \\
l_A &: 1 \times A \overset{\cong}{\longrightarrow} A \\
l_A &:= \pi_2,
\end{aligned}
$$

$$
\begin{array}{ccccc}
A & \quad & 1 \times A & \xrightarrow{\quad l_A \quad} & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \langle \mathrm{id}_1, f \rangle} & & \downarrow{\scriptstyle f} \\
B & & 1 \times B & \xrightarrow[\quad l_B \quad]{} & B
\end{array}
$$

$$
\begin{aligned}
r &: - \times 1 \overset{\cong}{\Longrightarrow} \mathsf{Id} \\
r_A &: A \times 1 \overset{\cong}{\longrightarrow} A \\
r_A &:= \pi_1.
\end{aligned}
$$

$$
\begin{array}{ccccc}
A & \quad & A \times 1 & \xrightarrow{\quad r_A \quad} & A \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \langle f, \mathrm{id}_1 \rangle} & & \downarrow{\scriptstyle f} \\
B & & B \times 1 & \xrightarrow[\quad r_B \quad]{} & B
\end{array}
$$

Natural Isomorphisms

Exercise 3

Verify that the families of arrows $s_{A,B}$, l_A and r_A, from the previous examples, are natural isomorphisms (textbook, Exercise 53).

Natural Isomorphisms

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Verify that the families of arrows $s_{A,B}$, l_A and r_A, from the previous examples, are natural isomorphisms (textbook, Exercise 53).

Remark

Because natural isomorphisms are a self-dual notion, we get also natural isomorphisms s, l and r for a category with binary coproducts and initial object.

Natural Transformations Between Hom-Functors

Natural Transformations Between Hom-Functors

Definition

Let $\mathcal{C}$ be a (locally small) category and let A and B be objects of $\mathcal{C}$. Recall the hom-functors

$$\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set} \qquad \text{(covariant)}$$

$$\mathcal{C}(-, B) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set} \qquad \text{(contravariant)}$$

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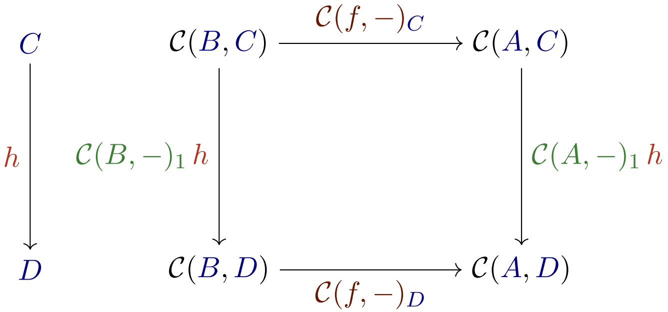
Natural Transformations Between Hom-Functors

Definition (continuation, first notation)

Let $f : A \to B$ in $\mathcal{C}$. We define the natural transformation $\mathcal{C}(f, -)$ and we show its naturality condition for $h : C \to D$ in $\mathcal{C}$.

$$\mathcal{C}(f, -) : \mathcal{C}(B, -) \Rightarrow \mathcal{C}(A, -)$$
$$\mathcal{C}(f, -)_C : \mathcal{C}(B, C) \to \mathcal{C}(A, C)$$
$$\mathcal{C}(f, -)_C\, g : \mathcal{C}(A, C)$$
$$\mathcal{C}(f, -)_C\, g := g \circ f$$



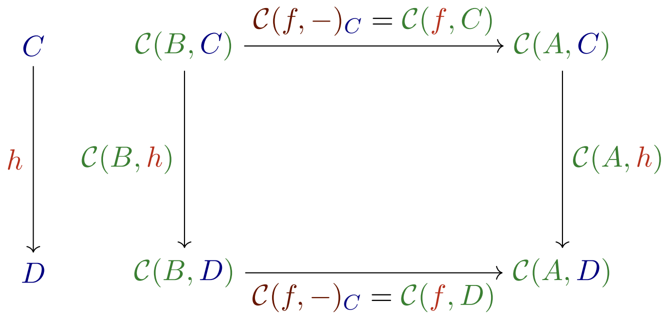
where $\mathcal{C}(f, -)_C = \mathcal{C}(-, C)_0\, f$.

Natural Transformations Between Hom-Functors

Definition (continuation, second notation)

Let $f : A \to B$ in $\mathcal{C}$. We define the natural transformation $\mathcal{C}(f,-)$ and we show its naturality condition for $h : C \to D$ in $\mathcal{C}$.

$$\mathcal{C}(f,-) : \mathcal{C}(B,-) \Rightarrow \mathcal{C}(A,-)$$
$$\mathcal{C}(f,-)_C : \mathcal{C}(B,C) \to \mathcal{C}(A,C)$$
$$\mathcal{C}(f,-)_C\, g : \mathcal{C}(A,C)$$
$$\mathcal{C}(f,-)_C\, g := g \circ f$$



Natural Transformations Between Hom-Functors

Exercise 4

Define the natural transformation $\mathcal{C}(-, f) : \mathcal{C}(-, A) \Rightarrow \mathcal{C}(-, B)$ and verify its naturality (textbook, Exercise 55).

Yoneda Lemma

Yoneda embedding

Let $\mathcal{C}$ be a locally small category and let A and B be objects of $\mathcal{C}$. For each natural transformation between hom-functors $\tau : \mathcal{C}(A, -) \to \mathcal{C}(B, -)$, there is a unique arrow $f : B \to A$ such that

$$\tau = \mathcal{C}(f, -).$$

Yoneda Lemma

Yoneda embedding

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$$\tau = \mathcal{C}(f, -).$$

Remark

The Yoneda lemma is a generalisation of the Yoneda embedding.

Compositions of Natural Transformations

Vertical Composition

Definition

Let $\mathcal{C}$ be a category, let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors, and let $\tau : F \Rightarrow G$ and $\mu : G \Rightarrow H$ be natural transformations. The **vertical composition** of μ and τ is the natural transformation $\mu \circ \tau$ defined by

$$\mu \circ \tau : F \Rightarrow H$$
$$(\mu \circ \tau)_A : F_0\, A \to H_0\, A := \mu_A \circ \tau_A, \quad \text{for all } A \text{ in } \mathcal{C}.$$

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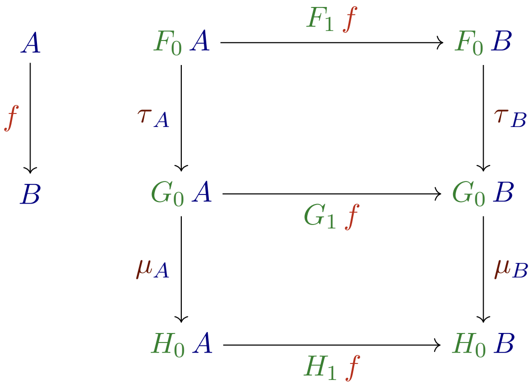
Vertical Composition

Definition (continuation)

That is, for all $f : A \to B$ in $\mathcal{C}$ the following diagram commutes.



$$\begin{pmatrix} (G_1\ f) \circ \tau_A = \tau_B \circ (F_1\ f) \\ (H_1\ f) \circ \mu_A = \mu_B \circ (G_1\ f) \\ (H_1\ f) \circ (\mu \circ \tau)_A = (\mu \circ \tau)_B \circ (F_1\ f) \end{pmatrix}$$

Vertical Composition

Exercise 5

Let $\mathcal{C}$ be a category, let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors, and let $\tau : F \Rightarrow G$ and $\mu : G \Rightarrow H$ be natural transformations. Show that $\mu \circ \tau : F \Rightarrow H$ is a natural transformation.

Functor Category

Functor Category

Introduction

Since we can define an associative composition of natural transformations and this composition has an identity natural transformation, we can define a category where the objects are functors and the arrows are natural transformations.

Functor Category

Definition

Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be an arbitrary category. The **functor category** $\text{Func}(\mathcal{C}, \mathcal{D})$ is defined by

(i) Objects: Functors $F : \mathcal{C} \to \mathcal{D}$.

(ii) Arrows: Natural transformations $\tau : F \Rightarrow G$.

Functor Category

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(iii) Composition of arrows: Vertical composition of natural transformations.

Functor Category

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(i) Objects: Functors $F : \mathcal{C} \to \mathcal{D}$.

(ii) Arrows: Natural transformations $\tau : F \Rightarrow G$.

(iii) Composition of arrows: Vertical composition of natural transformations.

(iv) Identity arrow

$$
\begin{aligned}
\mathsf{id}_F &: F \to F \\
(\mathsf{id}_F)_A &: F_0\, A \to F_0\, A \quad \text{for all } A \text{ in } \mathcal{C}, \\
(\mathsf{id}_F)_A &:= \mathrm{id}_{(F_0\, A)}.
\end{aligned}
$$

References

References

Abramsky, S. and Tzevelekos, N. (2011). Introduction to Categories and Categorical Logic. In: New Structures for Physics. Ed. by Coecke, B. Vol. 813. Lecture Notes in Physics. Springer, pp. 3–94. DOI: 10.1007/978-3-642-12821-9_1 (cit. on p. 2).

Awodey, S. [2006] (2010). Category Theory. 2nd ed. Vol. 52. Oxford Logic Guides. Oxford University Press (cit. on pp. 34, 35).

Mac Lane, S. [1971] (1998). Categories for the Working Mathematician. 2nd ed. Springer (cit. on pp. 5, 6).