Category Theory and Functional Programming
Appendix

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Preliminaries

Convention

The number assigned to chapters, examples, exercises, figures, pages, sections, and theorems on these slides correspond to the numbers assigned in the textbook [Abramsky and Tzevelekos 2011].

Outline

Monoids

Monoids

Definition

Let M be a set and let $(-) \cdot (-)$ be a binary relation on M and $1 \in M$. The structure $(M, \cdot, 1)$ is a **monoid** iff it satisfies

$$\forall x \forall y \forall z ((x \cdot y) \cdot z = x \cdot (y \cdot z)) \qquad \text{(associativity)}$$

$$\forall x (x \cdot 1 = x = 1 \cdot x) \qquad \text{(identity)}$$

Monoids

Example

The structure $(\mathbb{N}, +, 0)$ is a monoid.

Free Monoid

Definition

Let Σ be an alphabet (a set), let Σ^* be the set of strings over Σ including the empty string ε, and let $(-) \cdot (-)$ be the concatenation of strings. Then $(\Sigma^*, \cdot, \varepsilon)$ is the **free monoid** on the set Σ.

Monoid Homomorphisms

Definition

A **homomorphism** between monoids is a map between the domains of the monoids that preserves the monoid operation and the identity element.

Monoid Homomorphisms

Definition

A **homomorphism** between monoids is a map between the domains of the monoids that preserves the monoid operation and the identity element.

Let $(M, \cdot, 1_M)$ and $(N, *, 1_N)$ be two monoids. A homomorphism from $(M, \cdot, 1_M)$ to $(N, *, 1_N)$ is a function $h : M \to N$ such that for all x, y in M:

$$h\,(x \cdot y) = h\,x * h\,y,$$
$$h\,(1_M) = 1_N.$$

Groups

Groups

Definition

Let G be a set, $(-) \cdot (-)$ be a binary relation on G and $1 \in G$. The structure $(G, \cdot, 1)$ is a **group** iff it satisfies

$$\forall x \forall y \forall z ((x \cdot y) \cdot z = x \cdot (y \cdot z)) \qquad \text{(associativity)}$$

$$\forall x (x \cdot 1 = x = 1 \cdot x) \qquad \text{(identity)}$$

$$\forall x \exists x' (x \cdot x' = 1 = x' \cdot x) \qquad \text{(inverse)}$$

Groups

Example

The structure $(\mathbb{Z}, +, 0)$ is a group.

Groups

Example

The structure $(\mathbb{Z}, +, 0)$ is a group.

Example

The monoid $(\Sigma^*, \cdot, \varepsilon)$ is not a group.

Direct Product

Definition

Let $(G, *, 1_G)$ and $(H, \diamond, 1_H)$ be two groups. The **direct product** of G and H is the group $(G \times H, \cdot, (1_G, 1_H))$ where

$$\begin{aligned}(-) \cdot (-) &: G \times H \to G \times H \\ (g_1, h_1) \cdot (g_2, h_2) &:= (g_1 * g_2, h_1 \diamond h_2).\end{aligned}$$

Direct Product

Definition

Let $(G, *, 1_G)$ and $(H, \diamond, 1_H)$ be two groups. The **direct product** of G and H is the group $(G \times H, \cdot, (1_G, 1_H))$ where

$$\begin{aligned} (-) \cdot (-) &: G \times H \to G \times H \\ (g_1, h_1) \cdot (g_2, h_2) &:= (g_1 * g_2, h_1 \diamond h_2). \end{aligned}$$

Exercise 1

Show that the direct product of two groups is a group.

Algebraic Structures

Algebraic Structures

Definition

An **algebraic structure** on a set $A \neq \emptyset$ is essentially a collection of n-ary operations on A [Birkhoff 1946, 1987].

Algebraic Structures

Description

A **homomorphism** is a structure-preserving map between two algebraic structures.

Algebraic Structures

Definition

A homomorphism φ between two algebraic structures is [Cohn 1981]:

- a **monomorphism** if φ is an injection,
- an **epimorphism** if φ is a surjection,
- an **endomorphism** if φ is from an algebraic structure to itself,
- an **isomorphism** if φ is a bijection,
- an **automorphism** if φ is a bijective endomorphism.

Pre-orders

Pre-orders

Definition

Let P be a set and let $\preceq$ be a binary relation on P. The relation $\preceq$ is a **pre-order** (or **quasi-order**) iff it satisfies

$$\forall x(x \preceq x) \qquad \text{(reflexivity)}$$

$$\forall x \forall y \forall z(x \preceq y \preceq z \Rightarrow x \preceq z) \qquad \text{(transitivity)}$$

The pair $(P, \preceq)$ is a **pre-ordered set** (or **quasi-ordered set**).

Pre-orders

Example

The pair $(\mathbb{N}, \leq)$ is a pre-ordered set.

Pre-orders

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Example

The pair $(\{*\}, \{(*, *)\})$ is a pre-ordered set.

Pre-orders

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Example

The pair $(\{*\}, \{(*, *)\})$ is a pre-ordered set.

Question

Is the pair $(\emptyset, \emptyset)$ a pre-ordered set?

Pre-orders

Example

The pair $(\mathbb{N}, \leq)$ is a pre-ordered set.

Example

The pair $(\{*\}, \{(*, *)\})$ is a pre-ordered set.

Question

Is the pair $(\emptyset, \emptyset)$ a pre-ordered set?

Answer: Yes!

Pre-orders

Definition

Let $(S, \preceq_S)$ and $(T, \preceq_T)$ be two pre-ordered sets. A **homomorphism** from $(S, \preceq_S)$ to $(T, \preceq_T)$ is a function $h : S \to T$ such that, for all $x, y \in S$,

$$x \preceq_S y \quad \text{implies} \quad h\,x \preceq_T h\,y.$$

Pre-orders

Definition

Let $(S, \preceq_S)$ and $(T, \preceq_T)$ be two pre-ordered sets. A **homomorphism** from $(S, \preceq_S)$ to $(T, \preceq_T)$ is a function $h : S \to T$ such that, for all $x, y \in S$,

$$x \preceq_S y \quad \text{implies} \quad h\,x \preceq_T h\,y.$$

That is, a homomorphism from $(S, \preceq_S)$ to $(T, \preceq_T)$ is a monotone map $h : S \to T$.

Partial Orders

Partial Orders

Definition

Let P be a set and let $\preceq$ be a binary relation on P. The relation $\preceq$ is a **partial order** iff it satisfies

$$\forall x (x \preceq x) \qquad \text{(reflexivity)}$$

$$\forall x \forall y (x \preceq y \preceq x \to x = y) \qquad \text{(anti-symmetry)}$$

$$\forall x \forall y \forall z (x \preceq y \preceq z \to x \preceq z) \qquad \text{(transitivity)}$$

The pair $(P, \preceq)$ is a **partially ordered set** (or **poset**).

Partial Orders

Example

The pre-ordered sets $(\mathbb{N}, \leq)$, $(\emptyset, \emptyset)$ and $(\{*\}, \{(*, *)\})$ are posets.

Partial Orders

Question

Are pre-ordered sets which are not posets?

Partial Orders

Question

Are pre-ordered sets which are not posets?

Answer: Yes! The figure shows an example.

$$\begin{tikzcd} a \arrow[loop left] \arrow[r, bend left] & b \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

Partial Orders

Definition

Let $(S, \preceq_S)$ and $(T, \preceq_T)$ be two posets. A **homomorphism** from $(S, \preceq_S)$ to $(T, \preceq_T)$ is a function $h : S \to T$ such that, for all $x, y \in S$,

$$x \preceq_S y \quad \text{implies} \quad h\,x \preceq_T h\,y.$$

Partial Orders

Definition

Let $(S, \preceq_S)$ and $(T, \preceq_T)$ be two posets. An **order isomorphism** from $(S, \preceq_S)$ to $(T, \preceq_T)$ is a one-one correspondence $h : S \to T$ such that, for all $x, y \in S$,

$$x \preceq_S y \quad \text{iff} \quad h\,x \preceq_T h\,y.$$

Partial Orders

Definition

Let $(S, \preceq_S)$ and $(T, \preceq_T)$ be two posets. The **product of posets** S and T is the poset $(S \times T, \preceq)$ where the **product order** $\preceq$ is defined by:

For all $x_1, x_2 \in S$ and $y_1, y_2 \in T$,

$$(x_1, y_1) \preceq (x_2, y_2) \qquad \text{iff} \qquad x_1 \preceq_S x_2 \text{ and } y_1 \preceq_T y_2.$$

Relational Structures

Relational Structures

Definition

Let L be a signature of a relational structure consisting of function and relation symbols, and let A and B be two L-structures. A **homomorphism** from A to B is a mapping h from the domain of A to the domain of B such that[†]

(i) for each n-ary function symbol F in L,

$$h(F^A\, x_1\, \ldots\, x_n) = F^B\,(h\, x_1)\, \ldots\, (h\, x_n),$$

(ii) for each n-ary relation symbol R in L,

$$R^A(x_1, \ldots, x_n) \text{ implies } R^B(h\, x_1, \ldots, h\, x_n).$$

[†] From `https://en.wikipedia.org/wiki/Homomorphism`.

Topological Spaces

Topological Spaces

Definition

A **topology** on a set X is a collection τ of subsets of X such that

(i) $\emptyset$ and X are belong to τ,

(ii) the union of (finite or infinite) members of τ belongs to τ,

(iii) the intersection of finite members of τ belongs to τ.

The pair (X, τ) is a **topological space**.

Topological Spaces

Example

Let τ be the set of all open intervals in $\mathbb{R}$. The pair $(\mathbb{R}, \tau)$ is a topological space.

Topological Spaces

Example

Let τ be the set of all open intervals in $\mathbb{R}$. The pair $(\mathbb{R}, \tau)$ is a topological space.

Example

Let $\mathcal{P}\,S$ be the power set of the set S. The pair $(S, \mathcal{P}\,S)$ is a topological space.

Topological Spaces

Example

The pair $(\emptyset, \{\emptyset\})$ is a topological space.

Topological Spaces

Example

The pair $(\emptyset, \{\emptyset\})$ is a topological space.

Example

The pair $(\{*\}, \{\emptyset, \{*\}\})$ is a topological space.

Topological Spaces

Definition

Let (X, τ_X) and (Y, τ_Y) be topological spaces. A function $f : X \to Y$ is **continuous** iff for all $V \in \tau_Y$, $f^{-1}(V) \in \tau_X$.

Topological Spaces

Definition

Let (X, τ) be topological space. A **base** (or **basis**) for τ is a collection $B \subset \tau$ such that every open set is a union of elements of B.

Topological Spaces

Definition

Let (X, τ_X) and (Y, τ_Y) be topological spaces. The **product topological space** of X and Y is the topological space $(X \times Y, \tau_{X\times Y})$, where $\tau_{X\times Y}$ is the topology generated by the Cartesian product $U_X \times U_Y \subset X \times Y$ of open sets $U_x \subset X$ and $U_Y \subseteq Y$.

Category Theory

Remark

The following definition was adapted from [Mac Lane 1971].

Axiomatic Category Theory

Remark

The following definition was adapted from [Mac Lane 1971].

Definition

Axiomatic Category Theory is the following two-sorted first-order theory with equality:

- The sorts of the theory are $\mathrm{Obj}()$ (*objects*), denoted by $A, B, C, \ldots$, and $\mathrm{Ar}()$ (*arrows*), denoted by $f, g, h, \ldots$.

(continued on next slide)

Definition (continuation)

- The undefined terms (language) of the theory are the *function* symbols†

$$
\begin{aligned}
\mathrm{dom} &: \langle \mathrm{Ar}(), \mathrm{Obj}() \rangle && (\textit{domain}),\\
\mathrm{cod} &: \langle \mathrm{Ar}(), \mathrm{Obj}() \rangle && (\textit{codomain}),\\
\mathrm{id} &: \langle \mathrm{Obj}(), \mathrm{Ar}() \rangle && (\textit{identity arrow}),
\end{aligned}
$$

and the *relation* symbol

$$
\mathrm{comp} : \langle \mathrm{Ar}(), \mathrm{Ar}(), \mathrm{Ar}() \rangle \qquad (\textit{arrow composition}).
$$

(continued on next slide)

†The notation $\langle s_1, s_2, \dots, s_n \rangle$ denotes a sort in many-sorted logic. See, for example, [Enderton 2001].

Axiomatic Category Theory

Definition (continuation)

Notation. An arrow f with $\operatorname{dom} f = A$ and $\operatorname{cod} f = B$ is written $f : A \to B$.

Notation. The arrow $\mathrm{id}(A)$ is denoted id_A.

(continued on next slide)

Definition (continuation)

- Non-logical axioms
  (i) For all arrows f and g, if $f : A \to B$ and $g : B \to C$ then there exists an unique arrow $h : A \to C$, such as $\mathrm{comp}(f, g, h)$.

  Notation. If $\mathrm{comp}(f, g, h)$ then the arrow h is denoted $g \circ f$.

  (ii) For all arrows f, g and h, if $f : A \to B$, $g : B \to C$ and $h : C \to D$ then

  $$h \circ (g \circ f) = (h \circ g) \circ f.$$

  (iii) For all object A, $\mathrm{dom}(\mathrm{id}_A) = \mathrm{cod}(\mathrm{id}_A) = A$.

  (iv) For all arrow f, if $f : A \to B$ then

  $$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f.$$

Axiomatic Category Theory

Remark

In general, would be incorrect to define categories as *models* of the previous two-sorted theory because, because *set theory models* would not include *large* categories.

References

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